Quantum quasi-symmetric functions and Hecke algebras

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Abstract. The algebra of quasi-symmetric functions is known to describe the characters of the Hecke algebra $H_n(v)$ of type A_{n-1} at $v = 0$. We present a quantization of this algebra, defined in terms of filtrations of induced representations of the 0-Hecke algebra. We show that this q-deformed algebra admits a simple realization in terms of quantum polynomials. For generic values of q, the algebra of quantum quasi-symmetric functions is isomorphic to the one of noncommutative symmetric functions. This gives rise to a one parameter family of Hilbert space structures on the algebra of noncommutative symmetric functions, as well as to new interesting bases.

1. Introduction

It is well known that characters of the symmetric group S_n are encoded by symmetric functions, and this correspondence is the cornerstone of many computational methods in representation theory (cf. [15, 24]). The same correspondence works as well for the Hecke algebra $H_n(v)$ when v is neither 0 nor a root of unity (see [1]). Recently, it has been understood that certain generalizations of symmetric functions, originally introduced for different purposes, were the appropriate objects to encode the representation theory of the Hecke algebra at $v = 0$ [5, 12].

The first of these generalizations is the algebra of *quasi-symmetric functions*, introduced by Gessel [10] in his investigation of Kronecker products of certain representations of symmetric groups. The second one is the algebra of *noncommutative symmetric functions* [7], originally introduced with the aim of extending to the Gelfand-Retakh quasi-determinants [8, 9] the symmetric function interpretations of certain determinantal identities. Both algebras are endowed with natural structures of Hopf algebras. As shown by Malvenuto and Reutenauer [17] (see also [7]), these Hopf algebras are dual to each other. Moreover, both algebras have distinguished bases, in which the structure constants are nonnegative integers. This raised the question of a representation theoretical interpretation, and the answer was eventually found to be provided by Hecke algebras at $v = 0$ [5].

The need for two different Hopf algebras in the description of the representations of $H_n(0)$ comes from the fact that this algebra is not semisimple. If we decide to consider two finite dimensional $H_n(0)$ modules as equivalent whenever they have the same composition factors, the equivalence classes of all finitely generated modules of all 0-Hecke algebras, endowed with an appropriate induction product, build up the ring of quasi-symmetric functions. On the other hand, if we restrict to the class of finitely generated projective modules, with isomorphism as equivalence relation, we obtain by the same process the ring of noncommutative symmetric functions, and their duality can now be traced back to a general fact in representation theory (cf. [2]).

In the case where v is a nth root of unity, the corresponding rings can be respectively identified to the basic representation of the quantum affine algebra $U_q(\widehat{sl}_n)$ and to its dual, and the standard Fock space realization of these representations lead to the discovery of natural q-analogues of these representation rings [13, 14], the extra information provided by q being conjectured to describe natural filtrations of the modules.

In this paper, we describe a simple q-analogue of the algebra of quasi-symmetric functions, regarded as the ring of equivalence classes of finitely generated $H_n(0)$ modules. This q-analogue is defined by means of filtrations of induced modules, and involves a q-analogue of the shuffle product. This quantum shuffle, which is the simplest particular case of a construction of Rosso [20], is investigated in [3], where various connections were established, in particular with Greenberg's quon algebra [11, 6, 18]. For generic values of q, the algebra of quantum quasi-symmetric functions is isomorphic to the algebra of noncommutative symmetric functions. Moreover, it turns out that the algebra of quantum quasi-symmetric functions can be realized by means of quantum polynomials, following one of the specialization schemes proposed in [7] for noncommutative symmetric functions. This allows one to express the usual bases of noncommutative symmetric functions in terms of those of quasi-symmetric functions, leading to formulas which have no classical analogue. Also, this provides a natural family of Hilbert space structures on noncommutative symmetric functions, described by q-analogues of classical formulae.

2. The 0-Hecke algebra and its representations

The Hecke algebra $H_n(v)$ of type A_{n-1} is generated by $n-1$ elements $T_1, \ldots, T_{n-1}$ subject to the relations

$$T_i T_j = T_j T_i \qquad \text{for } |j-i| > 1 \tag{1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for } i < n-1 \tag{2}$$

$$T_i^2 = (v-1)T_i + v \tag{3}$$

For $v = 0$ this algebra is not semisimple, and it can be shown [19] that it has 2^{n-1} inequivalent irreducible representations. These representations, which are 1-dimensional, are defined as follows. For a subset $D \subseteq \{1, 2, \ldots, n-1\}$, set

$$\rho_D(T_i) = \begin{cases} -1 & \text{if } i \in D \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

Clearly, these formulae define a representation of $H_n(0)$ in the one dimensional space $\mathbf{C}$. For technical reasons, it is better to use as label for this representation the integer vector $I = C(D) = (i_1, \ldots, i_{r+1})$ defined for $D = \{d_1, \ldots, d_r\}$ by $i_k = d_k - d_{k-1}$, where we set $d_0 = 0$ and $d_{r+1} = n$. Thus, I is a composition of n, i.e. a positive integer vector with sum $|I| = n$. We will set $\varphi_I = \rho_D$, and the representation space will be denoted by $\mathbf{C}_I$. The subset D corresponding to the composition I is called the descent set of I, and is denoted by $\mathrm{Des}(I)$.

Let $QSym$ be the linear subspace of the polynomial ring $\mathbf{C}[x_1, x_2, \ldots]$ (in an infinite number of commuting variables) spanned by the elements (called quasi-monomial functions)

$$M_I = \sum_{k_1 < k_2 < \ldots < k_r} x_{k_1}^{i_1} x_{k_2}^{i_2} \cdots x_{k_r}^{i_r} \tag{5}$$

where I runs over all compositions. It turns out that $QSym$ is closed under the product of polynomials. It is called the algebra of quasi-symmetric functions [10]. An important distinguished basis of $QSym$, the quasi-ribbon functions, is defined by

$$F_I = \sum_{J \succeq I} M_J \tag{6}$$

where $J \succeq I$ means that J is finer than I, i.e. that $\mathrm{Des}(I) \subseteq \mathrm{Des}(J)$, e.g. $(1, 2, 2, 1, 1, 3) \succeq (3, 3, 4)$.

The link between quasi-symmetric functions and 0-Hecke algebras can now be described as follows [5]. First of all, the Specht modules $V_\lambda(v)$ of the generic Hecke algebra $H_n(v)$ still make sense for $v = 0$ but are no longer irreducible. Let $d_{\lambda I}$ be the multiplicity of $\mathbf{C}_I$ as a composition factor of $V_\lambda(0)$. It follows from Carter's description of the decomposition matrix [1] that these numbers are given by

$$s_\lambda = \sum_{|I|=n} d_{\lambda I} F_I \tag{7}$$

where s_λ is a Schur function, regarded as a quasi-symmetric function. Moreover, if ones defines a Frobenius map $\mathcal{F}$ sending the class of the module $\mathbf{C}_I$ to the quasi-symmetric function F_I, one can show that this map is compatible with outer tensor products as in the case of symmetric groups. That is, if one defines, for a $H_m(0)$-module M and a $H_n(0)$-module N, the outer product $M \hat{\otimes} N$ as the induced representation

$$M \hat{\otimes} N = (M \otimes N) \uparrow_{H_m(0) \otimes H_n(0)}^{H_{m+n}(0)} \tag{8}$$

then

$$\mathcal{F}(M\widehat{\otimes}N) = \mathcal{F}(M)\mathcal{F}(N) \ . \tag{9}$$

The algebra of noncommutative symmetric functions, although defined as an abstract algebra, can be concretely realized as follows [7]. Let $A = \{a_1, a_2, \dots\}$ be an infinite set of noncommuting variables. The complete homogeneous noncommutative symmetric functions $S_n(A)$ and the elementary symmetric functions $\Lambda_n(A)$ are defined by the generating functions

$$\sigma_t(A) = \sum_{n\geq 0} t^n S_n(A) = \overrightarrow{\prod_{k\geq 1}} (1 - ta_k)^{-1} \tag{10}$$

$$\lambda_{-t}(A) = \sum_{n\geq 0} (-t)^n \Lambda_n(A) = \sigma_t(A)^{-1} = \overleftarrow{\prod_{k\geq 1}} (1 - ta_k) \tag{11}$$

where t is a variable commuting with the a_i. The algebra **Sym** of noncommutative symmetric functions is freely generated by either the S_i or the Λ_i. For a composition $I = (i_1, \dots, i_r)$ one defines the elements $S^I = S_{i_1} \cdots S_{i_r}$, $\Lambda^I = \Lambda_{i_1} \cdots \Lambda_{i_r}$, and the ribbon Schur functions

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S_J \tag{12}$$

where $\ell(I) = r$ is the length (number of parts) of a composition I. In terms of the 0-Hecke algebra, ribbon Schur functions correspond to isomorphism classes of principal projective indecomposable modules, which explains that the two Hopf algebras $QSym$ and **Sym** can be put in duality by means of the pairing

$$\langle F_I \, , \, R_J \rangle = \delta_{IJ} \tag{13}$$

(for which one also has $\langle M_I \, , \, S^J \rangle = \delta_{IJ}$).

3. Quantum quasi-symmetric functions

The product $F_I F_J$, which corresponds to the outer tensor product of two irreducible $H_n(0)$-modules, can be described in terms of shuffles of permutations [16]. Permutations can be considered as words on the letters $1, 2, \dots, n$, and, in general, the shuffle product $u \sqcup\!\sqcup v$ of two words on some alphabet A can be defined by the recursive formula

$$\text{if } u = au' \text{ and } v = bv',\ a, b \in A, \ \ u \sqcup\!\sqcup v = a(u' \sqcup\!\sqcup v) + b(u \sqcup\!\sqcup v') \tag{14}$$

with the initial condition $u \sqcup\!\sqcup \epsilon = \epsilon \sqcup\!\sqcup u = u$, ϵ being the empty word.

We also need the notions of descent set and descent composition of a permutation. One says that i is a descent of $\sigma \in \mathrm{S}_n$ if $\sigma(i) > \sigma(i+1)$. The descent set of σ will be denoted by $\mathrm{Des}(\sigma)$. The composition associated to this set by the process described in the preceding section is denoted by $C(\sigma)$ and is called the descent composition of σ.

Now, to multiply F_I and F_J, where $|I| = n$ and $|J| = m$, take any permutation u of $1, \dots, n$ such that $C(u) = I$ and any permutation v of $n+1, \dots, n+m$ such that $C(v) = J$. Then the shuffle of the two words u and v is a sum of permutations of $1, \dots, n+m$

$$u \sqcup\!\sqcup v = \sum_{w \in \mathrm{S}_{m+n}} c_w w \tag{15}$$

and the product is given by

$$F_I F_J = \sum_{w \in \mathrm{S}_{m+n}} c_w F_{C(w)} \ . \tag{16}$$

There exists a q-analogue of the shuffle product, which is known to be related to the representation theory of $H_n(0)$ [5, 3]. This quantum shuffle, which is the simplest case of Rosso's construction [20] (it corresponds to the choice of a scalar matrix as solution of the Yang-Baxter equation), is defined by

$$\text{if } u = au' \text{ and } v = bv',\ a, b \in A,\ \ u \sqcup\!\sqcup_q v = a(u' \sqcup\!\sqcup_q v) + q^{|u|} b(u \sqcup\!\sqcup_q v') \tag{17}$$

where $|u|$ is the length of u. It can be shown that this operation is associative, and that when q is not a root of unity, the q-shuffle algebra is isomorphic to the concatenation algebra, which corresponds to the case $q = 0$. This follows from Zagier's formula for the determinant of the operator $U_n(q) = \sum_{\sigma \in \mathrm{S}_n} q^{\ell(\sigma)} \sigma$ of the regular representation of S_n [25], and has to do with the fact that the quonic Fock space is the same for all $q \in (-1, 1)$ [22].

The representation theoretical interpretation of the q-shuffle is as follows. The induced representation $\mathbb{C}_I \widehat{\otimes} \mathbb{C}_J$ is generated by the vector $|0\rangle = 1 \otimes 1 \in \mathbb{C}_I \otimes \mathbb{C}_J$. There is a filtration of this module whose k-th slice M_k is spanned by the elements $T_\sigma |0\rangle$ for permutations σ of length k. Now, if one computes the product $F_I F_J$ by using the q-shuffle instead of the ordinary one in formula (16), the coefficient of $q^k F_H$ in the result is the multiplicity of F_H at level k of the filtration.

This suggest to define the algebra $QSym_q$ of quantum quasi-symmetric functions as the algebra with generators F_I and multiplication rule

$$F_I F_J = \sum_w c_w(q) F_{C(w)} \tag{18}$$

for permutations u and v as above, $c_w(q) \ = \ < w | u \sqcup\!\sqcup_q v >$ being the coefficient of w in $u \sqcup\!\sqcup_q v$. We also define the basis (M_I), of $QSym_q$ as well as analogues of other relevant bases by the same relations as in the classical case.

For generic values of q, $QSym_q$ is freely generated by the one-part quasi-ribbons F_n, or as well by the power-sums M_n, or any sequence corresponding to a free set of generators of the algebra of symmetric functions in the classical case. This means that if we define for a composition $I = (i_1, \dots, i_r)$

$$F^I = F_{i_1} F_{i_2} \cdots F_{i_r} \ \text{ and } \ M^I = M_{i_1} M_{i_2} \cdots M_{i_r} \ \in \ QSym_q \tag{19}$$

then the F^I (resp. the M^I) form a basis of $QSym_q$ (this is clearly not true for $q = 1$, as in this case these elements are symmetric functions). The easiest way to see this is to take as generators $E_n = F_{1^n}$ (corresponding to elementary symmetric functions). Indeed, $E^I = E_{i_1}E_{i_2}\cdots E_{i_r} = F_{\overline{I}^\sim} + O(q)$ where $I^\sim$ is the conjugate composition of I and $\overline{I}$ the mirror image composition $(i_r,\ldots,i_1)$. Thus, the map $F_I \mapsto E_{\overline{I}^\sim}$ is invertible, since its matrix is of the form $1 + O(q)$.

Thus, for generic q, $QSym_q$ is isomorphic to the algebra of noncommutative symmetric functions, and the natural correspondence is to identify S_n with F_n, since both elements are noncommutative analogues of the complete homogeneous symmetric functions h_n. Thus, we define a ring isomorphism $f \mapsto \hat{f}$ from **Sym** into $QSym_q$ by $\hat{S}_n = F_n$. We have then a realization of **Sym** in a space which is a q-analogue of its dual, and we can now define a scalar product on this space by setting

$$(F_I|\hat{R}_J) = \delta_{IJ} \tag{20}$$

in accordance with the duality formula (13). It follows from Zagier's work on the quon algebra [25] that this scalar product degenerates when q is a root of unity, and in particular in the classical limit $q = 1$. However, there are other singular values of the parameter, but we do not know how to characterize them. The first singular values of q in the complex plane are plotted below.

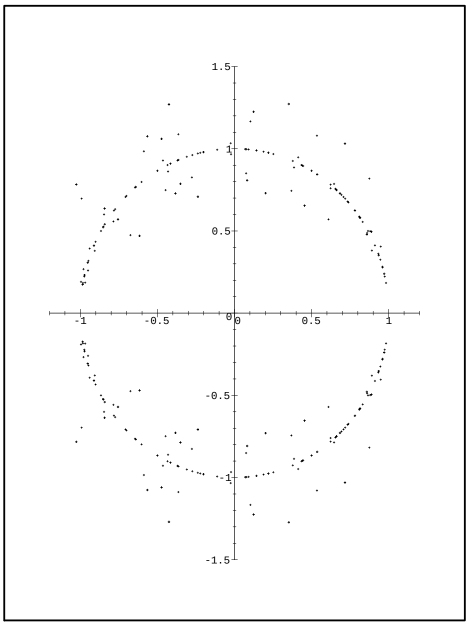


Figure 1. Singular values of q for $n \leq 7$

4. Quantum quasi-symmetric functions as q-polynomials

Let $\mathbb{C}_q[X] = \mathbb{C}_q[x_1, x_2, \ldots]$ be the associative algebra generated by an infinite sequence of elements x_i subject to the commutation relations

$$\text{for } j > i, \ \ x_j x_i = q x_i x_j \ . \tag{21}$$

Let $\mathbf{Sym}(X)$ be the subalgebra of $\mathbb{C}_q[X]$ generated by the specialization $a_i \to x_i$ of the noncommutative symmetric functions defined by formulas (10) or (11). We will prove that $\mathbf{Sym}(X)$ is isomorphic to $QSym_q$, the correspondence being given by

$$M_I \longleftrightarrow \overline{M}_I = \sum_{j_1 < \cdots < j_r} x_{j_1}^{i_1} \ldots x_{j_r}^{i_r} \ . \tag{22}$$

That is, if ones defines

$$\overline{F}_I = \sum_{J \succeq I} \overline{M}_J \ , \tag{23}$$

one has for u a permutation of $1, \ldots, n$ and v a permutation of $n+1, \ldots, n+m$

$$\overline{F}_{C(u)} \overline{F}_{C(v)} = \sum_w < w | u \sqcup\!\sqcup_q v > \overline{F}_{C(w)} \tag{24}$$

where $< w | u \sqcup\!\sqcup_q v >$ is the coefficient of the word w in the q-shuffle $u \sqcup\!\sqcup_q v$. As $\bar{F}_n = S_n(X)$, this will be sufficient to prove our assertion. To establish (24), we need to recall some results from the theory of partially ordered sets (posets, cf. [21]). Let P be a partial order on $[n] = \{1, \ldots, n\}$. We write $<_P$ for the partial order P and $<$ for the usual order on $[n]$. We denote by $L(P)$ the set of standard words w on $[n]$ with $|w| = n$ such that if $x <_P y$ then x occurs on the left of y in w. A P-partition is a function $f \ : \ P \to X$ such that

- $i <_P j$ then $f(i) \le f(j)$,
- $i <_P j$ and $i > j$ then $f(i) < f(j)$,

the ordering on X being the natural one ($x_i < x_j$ for $i < j$). The set of all P-partitions is denoted by $A(P)$. Generalizing a construction of Gessel [10], we define the q-generating function of a poset P as

$$\Gamma_q(P) = \sum_{f \in A(P)} f(1) f(2) \ldots f(n) \qquad \in \mathbb{C}_q[X] \tag{25}$$

with $f(i) \in X$. To a standard word $w = w_1 w_2 \ldots w_n$ on $[n]$, one associates the poset $P(w)$ defined by $w_1 <_{P(w)} w_2 <_{P(w)} \cdots <_{P(w)} w_n$. Then, one can check that the q-generating function of $P(w)$ is given by

$$\Gamma_q(P(w)) = q^{l(w)} \overline{F}_{C(w)} \tag{26}$$

where $l(w)$ is the number of inversions of w. The q-generating function for $(P, <_P)$ is therefore

$$\Gamma_q(P) = \sum_{w \in L(P)} q^{l(w)} \overline{F}_{C(w)} \ . \tag{27}$$

For $q=1$, we obtain the classical generating function of P [10, 16]. A consequence of (25) is

$$\Gamma_q(P_1 \sqcup P_2) = \Gamma_q(P_1)\Gamma_q(P_2) \tag{28}$$

where P_1 is a poset on $\{1,\ldots,n\}$, P_2 a poset on $\{n+1,\ldots,n+m\}$ and $P = P_1 \sqcup P_2$ is defined as the poset on $\{1,\ldots,n+m\}$ for which $i <_P j$ iff $i <_{P_1} j$ or $i <_{P_2} j$ (this follows from (25)). Now, we argue as in [16]. Since P_1 is disjoint from P_2, the map $A(P_1 \sqcup P_2) \to A(P_1) \times A(P_2)$ given by $f \to (f|P_1, f|P_2)$ is a bijection, so that

$$\begin{aligned}
\Gamma_q(P_1 \sqcup P_2) &= \sum_{f\in A(P_1\sqcup P_2)} f(1)\ldots f(n)f(n+1)\ldots f(n+m) \\
&= \sum_{\substack{q\in A(P_1)\\ h\in A(P_2)}} g(1)\ldots g(n)h(1)\ldots h(m) \\
&= \Gamma_q(P_1)\Gamma_q(P_2)\ .
\end{aligned}$$

To complete the proof, we need the following property [16]. Suppose P_1, P_2 as above, then

$$L(P_1 \sqcup P_2) = L(P_1) ⧢ L(P_2)\ . \tag{29}$$

We are now in position to conclude. Let u be a standard word on $1\ldots n$ and v be a standard word on $n+1\ldots n+m$. Then, applying successively (26), (28), (27) and (29), we obtain

$$\begin{aligned}
\overline{F}_{C(u)}\overline{F}_{C(v)} &= q^{-l(u)-l(v)}\Gamma_q(P(u))\Gamma_q(P(v)) \\
&= q^{-l(u)-l(v)}\Gamma_q(P(u)\sqcup P(v)) \\
&= q^{-l(u)-l(v)} \sum_{w\in L(P(u)\sqcup P(v))} q^{l(w)}\overline{F}_{C(w)} \\
&= q^{-l(u)-l(v)} \sum_{w\in u ⧢ v} q^{l(w)}\overline{F}_{C(w)}
\end{aligned} \tag{30}$$

We recall that given two standard words u, v as above

$$u ⧢_q v = q^{-l(u)-l(v)} \sum_{w\in u⧢ v} q^{l(w)} w\ . \tag{31}$$

Combining (30) and (31), we obtain

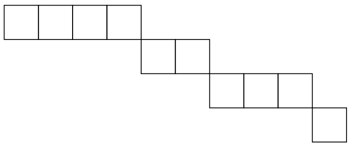
$$\overline{F}_{C(u)}\overline{F}_{C(v)} = \sum_w < w|u ⧢_q v > \overline{F}_{C(w)} \tag{32}$$

which establishes the formula.

5. Some formulae

The fact that noncommutative symmetric functions can be realized in $QSym_q$ leads to the possibility of expressing the usual bases of noncommutative symmetric functions in terms of quasi-symmetric functions (and conversely), which is clearly meaningless in the classical case.

We first recall some results. Let $I = (i_1, \ldots, i_r)$ be a composition, we shall denote by $[I] = i_1 \times i_2 \times i_3 \times \cdots \times i_r$ the *skew diagram* obtained by juxtaposing corner to corner rows of successive lenghts $i_1, i_2, \ldots, i_r$. For instance, $[4, 2, 3, 1]$ is the diagram indicated by the configuration of boxes illustrated below.



Let $f \to f^\vee$ be the linear involution of $\mathbb{C}\mathrm{S}_n$ defined by $\sigma^\vee = \sigma^{-1}$. One says that a permutation fits $[I]$ if its descent set is contained in $\mathrm{Des}(I)$. Then one can check that

$$\sum_{\mathrm{Des}(\sigma) \subseteq \mathrm{Des}(I)} \sigma^{-1} = V_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup V_r \tag{33}$$

where $V_1 = 1 \ldots i_1$, $V_2 = i_1{+}1 \cdots i_1{+}i_2$, $\ldots$,$V_r = i_1{+}i_2{+}\cdots{+}i_{r-1}{+}1 \ldots i_1{+}i_2{+}\cdots{+}i_r$. For instance,

$$12 \sqcup\!\sqcup 34 = 1234 + 1324 + 1342 + 3124 + 3142 + 3412\ , \tag{34}$$

and

$$(12 \sqcup\!\sqcup 34)^\vee = \begin{array}{ll}\boxed{1}\boxed{2} & \\ & \boxed{3}\boxed{4}\end{array} + \begin{array}{ll}\boxed{1}\boxed{3} & \\ & \boxed{2}\boxed{4}\end{array} + \begin{array}{ll}\boxed{2}\boxed{3} & \\ & \boxed{1}\boxed{4}\end{array} + $$
$$\begin{array}{ll}\boxed{1}\boxed{4} & \\ & \boxed{2}\boxed{3}\end{array} + \begin{array}{ll}\boxed{2}\boxed{4} & \\ & \boxed{1}\boxed{3}\end{array} + \begin{array}{ll}\boxed{3}\boxed{4} & \\ & \boxed{1}\boxed{2}\end{array}\ ,$$

i.e. we obtain the set of permutations that fit the skew diagram associated with the composition $(2, 2)$.

From (33), it follows that

$$F^I = \sum_J \left(\sum_{\substack{C(\sigma) \preceq I \\ C(\sigma^{-1}) = J}} q^{l(\sigma)} \right) F_J\ . \tag{35}$$

We may now express the ribbon Schur functions on the quantum quasi-ribbons. The correspondence $\hat{S}_n = F_n$ gives

$$\hat{S}^I = \sum_J \left(\sum_{\substack{C(\sigma) \preceq I \\ C(\sigma^{-1}) = J}} q^{l(\sigma)} \right) F_J\ . \tag{36}$$

Let us now find the corresponding formula for $\hat{R}_I$. Combining (36) and (12) yields

$$\hat{R}_I = \sum_K \underbrace{(\sum_{\substack{J \preceq I \\ C(\sigma) \preceq J \\ C(\sigma^{-1}) = K}} (-1)^{l(I)+l(J)} q^{l(\sigma)})}_{B_K} F_K \ , \tag{37}$$

and B_K can be rewritten as

$$B_K = \sum_\sigma q^{l(\sigma)} (-1)^{l(I)} (\sum_{\substack{C(\sigma) \preceq J \\ J \preceq I}} (-1)^{l(J)}) \ .$$

We consider two cases:

- $C(\sigma) \neq I$. Then

$$\begin{aligned} \sum_{C(\sigma) \preceq J \preceq I} (-1)^{l(J)} &= \sum_{\mathrm{Des}\,(\sigma) \subseteq \mathrm{Des}\,(J) \subseteq \mathrm{Des}\,(I)} (-1)^{|\mathrm{Des}\,(J)|} \\ &= \sum_{i=0}^{|\mathrm{Des}\,(I)|} (-1)^i \binom{|\mathrm{Des}\,(I)|}{i} \\ &= 0 \end{aligned}$$

- $C(\sigma) = I$. It follows that

$$(-1)^{l(I)} \sum_{C(\sigma) \preceq J \preceq I} (-1)^{l(J)} = 1 \ .$$

Thus

$$B_K = \sum_{C(\sigma) = I} q^{l(\sigma)}$$

which establishes the formula

$$\hat{R}_I = \sum_J \left(\sum_{\substack{C(\sigma) = I \\ C(\sigma^{-1}) = J}} q^{\ell(\sigma)} \right) F_J \ . \tag{38}$$

From this, one deduces the following scalar product, which is the length q-analogue of the classical formula for the scalar product of two ordinary (commutative) ribbon Schur functions [10]

$$(\hat{R}_I | \hat{R}_J) = \sum_{C(\sigma) = I, \ C(\sigma^{-1}) = J} q^{\ell(\sigma)} \ . \tag{39}$$

These polynomials are also q-analogues of the Cartan invariants of $H_n(0)$. A representation theoretical interpretation will be given in the forthcoming section.

Also, using the following formula for power-sums of the first kind:

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k R_{(1^k, n-k)} \ , \tag{40}$$

one finds

$$\hat{\Psi}_n = \sum_{|I|=n} (-1)^{l(I)-1} q^{\mathrm{maj}\,(I)} F_I \tag{41}$$

where maj (I) the sum of its descent set (major index). The proof is as follows. For J a subset of $\{1, 2, \ldots, n-1\}$, we say that a standard hook tableau has type J if $i \in J$ if and only if i occurs in a lower row of the tableau than $i+1$ [1].
Then given a subset $D \subset \{1, \ldots, n-1\}$, there exists exactly one standard hook tableau such that T is of type D. Indeed, if $D = \{d_1, d_2, \ldots, d_k\}$ then the column part of T must be

$$\begin{array}{|c|}\hline d_k+1 \\ \hline d_{k-1}+1 \\ \hline \vdots \\ \hline d_1+1 \\ \hline \end{array}$$

and T is of shape $(n-k, 1^k)$. For instance, for $D = \{2, 5, 8, 9, 13\} \subset \{1, \ldots, 14\}$ we associate the standard hook

$$\begin{array}{|c|cccccccc}\cline{1-1} 14 \\ \cline{1-1} 10 \\ \cline{1-1} 9 \\ \cline{1-1} 6 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{11} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} & \multicolumn{1}{c|}{15} \\ \hline \end{array}$$

We thus get

$$\begin{aligned} R_{1^k, n-k} &= \sum_{|D|=k} q^{d_1+d_2+\cdots+d_k} F_{C(D)} \\ &= \sum_{l(I)=k+1} q^{\mathrm{maj}\,(I)} F_I \ . \end{aligned}$$

To complete the proof, we recall that for a composition I of n, $l(I) + l(I^{\sim}) = n+1$. This clearly implies (41).

The remainder of this section is devoted to the formulation of the multiplication rule of quantum quasi-monomial functions. We consider the operator $\odot_q$ defined as follows:

$$\begin{aligned} au \odot_q bv &= a(u \odot_q bv) + q^{||au||b} b(au \odot_q v) + q^{||u||b}(a+b)(u \odot_q v) \ , \\ a \odot_q \epsilon &= \epsilon \odot_q a = a \ , \end{aligned}$$

for $a, b \in \mathbb{N}$ and u, v two compositions regarded as words on the alphabet $\mathbb{N}^*$.
Then given two compositions I and J, we have:

$$M_I M_J = \sum_K < K \mid I \odot_q J > M_K$$

For example:

$$\begin{aligned} M_{21} * M_{12} = \ & (q+1)\, M_{2112} + q^3 M_{2121} + q M_{213} + M_{222} \\ & + q^3 M_{1212} + \left(q^5 + q^9\right) M_{1221} + q^3 M_{123} \\ & + q^5 M_{141} + q M_{312} + q^3 M_{321} + q M_{33} \end{aligned}$$

6. Interpretation of the q-Cartan invariants of $H_n(0)$

Norton obtained a description of the indecomposable projective modules over $H_n(0)$ [19]. For a subset A of $\{1,\ldots,n-1\}$, we define $\Box_A$ and ∇_A in $H_n(0)$ by

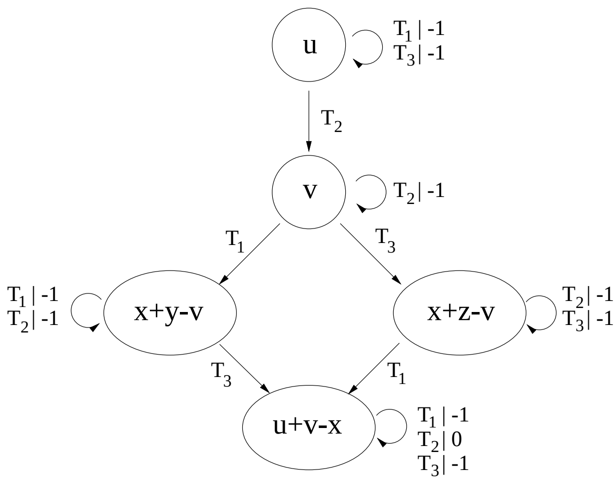
$$\Box_A = \sum_{w \in A} T_w, \ \nabla_A = T_{w_A} \tag{42}$$

where w_A is the maximal permutation of the subgroup generated by $\{\sigma_a | a \in A\}$. For a composition I of n, we set

$$\eta_I = \Box_{\overline{A}} \nabla_A \tag{43}$$

where $A = \mathrm{Des}\,(I)$ and $\overline{A} = \{1,\ldots,n-1\} \setminus \mathrm{Des}\,(I)$. The projective $H_n(0)$-module $\mathbf{M}_I$ is realized by the left ideal $H_n(0)\eta_I$ [1, 19]. As shown in [12], projective modules can always be described by a graph, i.e. possess a basis $v_1, v_2, \ldots, v_n$ of $\mathbf{M}_I$ such that $T_i v_j$ can be only $0, -v_j$ or another v_k.

Example 6.1. Let $I = (1,2,1)$, $A = \mathrm{Des}(I) = \{1,3\}$ and $\overline{A} = \{2\}$. Applying (43), we obtain $\eta_{121} = (1+T_2)T_1T_3$. The module $\mathbf{M}_{121}$ can be described by the following "automaton". An arrow indexed by T_i going from f to g means $T_i f = g$ and a loop on the vertex f indexed by $T_i|\epsilon$ (with $\epsilon = 0$ or $\epsilon = \pm 1$) means $T_i f = \epsilon f$:

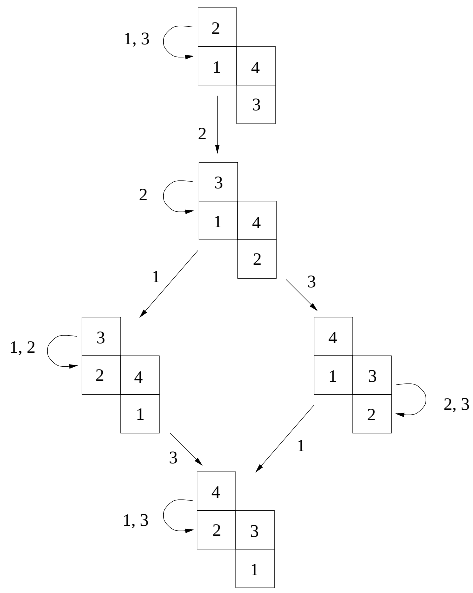


Here x $= \eta_{121}$, y $= T_1$x, z $= T_3$x, u $= T_3 T_1$x, v $= T_2 T_3 T_1$x.

$\mathbf{M}_I$ can also be realized as $H_n(0)\nu_I$, where $\nu_I = \nabla_A \square_{\overline{A}}$ [3]. With this choice of the generator ν_I, $\mathbf{M}_I$ has for basis $\{e_\sigma = T_\sigma \square_{\overline{A}} \mid \mathrm{Des}\,(\sigma) = \mathrm{Des}\,(I)\}$, and clearly, this satisfies

$$T_i e_\sigma = \begin{cases} e_{\sigma_i \sigma} & \text{if } l(\sigma_i\sigma) > l(\sigma) \text{ and } \mathrm{Des}\,(\sigma_i\sigma) = \mathrm{Des}\,(\sigma) \\ -e_\sigma & \text{if } l(\sigma_i\sigma) < l(\sigma) \\ 0 & \text{if } \mathrm{Des}\,(\sigma_i\sigma) \not\subseteq \mathrm{Des}\,(\sigma) \end{cases} \tag{44}$$

Example 6.2. Let $I = (1, 2, 1)$. Applying (44), we see that the generator of the module is indexed by the permutation obtained by filling the columns of the skew Young diagram of ribbon shape I from bottom to top and from left to right with the numbers $1, 2, \ldots, n$. The action of the generators of $H_4(0)$ on the basis (e_σ) of $\mathbf{M}_{121}$ is described by the following graph. An arrow indexed by i going from σ to σ' means $\sigma' = \sigma_i\sigma$ and a loop on a vertex σ is indexed by the type of the permutation σ.



This shows that these modules have natural filtrations, given by the distance to the initial vertex in the graph. Let $H_n^{(k)} = \oplus_{l(w)\geq k} \mathbb{C} T_w$ be the length filtration of $H_n(0)$ and $\mathbf{M}_I^{(k)} = H_n^{(k)}\nu_I$. The $\mathbf{M}_I^{(k)}$ are submodules of $\mathbf{M}_I$, of which they form a filtration. Then the graded characteristic of $\mathbf{M}_I$ is defined by

$$\mathcal{F}_q(\mathbf{M}_I) = \sum_k q^k \mathcal{F}(\mathbf{M}_I^{(k)}/\mathbf{M}_I^{(k+1)}) \tag{45}$$

and we have

$$\hat{R}_I = q^{l(w_A)} \mathcal{F}_q(\mathbf{M}_I) \ , \tag{46}$$

so that

$$\mathcal{F}_q(\mathbf{M}_I) = q^{-l(w_A)} \hat{R}_I$$

$$= q^{-l(w_A)} \sum_J \left(\sum_{\substack{C(\sigma)=I \\ C(\sigma^{-1})=J}} q^{l(\sigma)} \right) F_I .$$

Thus, the q-Cartan invariants describe the filtrations of the principal indecomposable projective modules.

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